# Exercise 6

Consider the two simple closed contours shown in Fig. 111 and obtained by dividing into two pieces the annulus formed by the circles $C_\rho$ and $C_R$ in Fig. 110 (Sec. 91). The legs $L$ and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, $\Gamma_\rho$ and $\gamma_\rho$ are the indicated portions of $C_\rho$, while $\Gamma_R$ and $\gamma_R$ make up $C_R$.

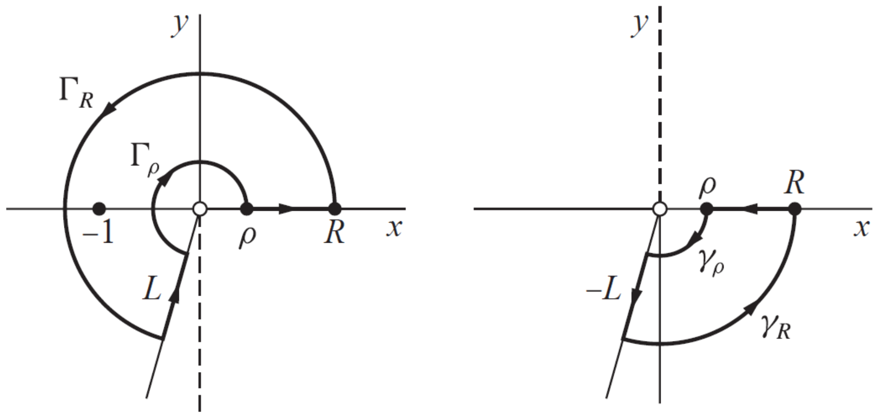


**FIGURE 111**

($a$) Show how it follows from Cauchy's residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{-a}/(z+1)$ is integrated around the closed contour on the left in Fig. 111,

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z).$$

($b$) Apply the Cauchy-Goursat theorem to the branch

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0,\ \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

of $z^{-a}/(z+1)$, integrated around the closed contour on the right in Fig. 111, to show that

$$-\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0.$$

($c$) Point out why, in the last lines in parts ($a$) and ($b$), the branches $f_1(z)$ and $f_2(z)$ of $z^{-a}/(z+1)$ can be replaced by the branch

$$f(z) = \frac{z^{-a}}{z+1} \qquad (|z| > 0,\ 0 < \arg z < 2\pi).$$

Then, by adding corresponding sides of those two lines, derive equation (3), Sec. 91, which was obtained only formally there.

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**Solution**

**Part (a)**

According to Cauchy's residue theorem, the integral of $f_1(z) = z^{-a}/(z+1)$ around the closed contour on the left is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_{C_\Gamma} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} \frac{z^{-a}}{z+1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{\Gamma_x} f_1(z)\,dz + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z),$$

where $\Gamma_x$ is the radial arc on the $x$-axis. The parameterization for it is as follows.

$$\Gamma_x : \quad z = re^{i0}, \qquad r = \rho \quad \to \quad r = R$$

As a result,

$$\int_\rho^R \frac{(re^{i0})^{-a}}{re^{i0}+1}(dr\, e^{i0}) + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z)$$

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z). \tag{1}$$

**Part (b)**

According to the Cauchy-Goursat theorem, the integral of $f_2(z) = z^{-a}/(z+1)$ around the closed contour on the right is equal to zero because there are no enclosed singularities.

$$\oint_{C_\gamma} f_2(z)\,dz = 0$$

This closed loop integral is the sum of four integrals, one over each arc in the loop. Note that because $\theta_0$ lies between $\pi$ and $3\pi/2$, the parameterization of $L$ is the same as in part (a) but with a minus sign.

$$\int_{\gamma_x} f_2(z)\,dz + \int_{\gamma_\rho} f_2(z)\,dz + \int_{-L} f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0,$$

where $\gamma_x$ is the radial arc on the $x$-axis. The parameterization for it is as follows.

$$\gamma_x : \quad z = re^{i2\pi}, \qquad r = R \quad \to \quad r = \rho$$

As a result,

$$\int_R^\rho \frac{(re^{i2\pi})^{-a}}{re^{i2\pi}+1}(dr\, e^{i2\pi}) + \int_{\gamma_\rho} f_2(z)\,dz + \int_{-L} f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0$$

$$-\int_\rho^R \frac{r^{-a}e^{-i2\pi a}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0. \tag{2}$$

**Part (c)**

The parameterizations for $L$, $\Gamma_\rho$, $\Gamma_R$, $\gamma_\rho$, and $\gamma_R$ can be written only with angles between 0 and $2\pi$, so the branch cut

$$f(z) = \frac{z^{-a}}{z+1} \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

can be used instead of $f_1(z)$ and $f_2(z)$. Adding both sides of equations (1) and (2) together, replacing $f_1(z)$ and $f_2(z)$ with $f(z)$, we obtain

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f(z)\,dz + \int_L f(z)\,dz + \int_{\Gamma_\rho} f(z)\,dz - \int_\rho^R \frac{r^{-a}e^{-i2\pi a}}{r+1}\,dr \\ + \int_{\gamma_\rho} f(z)\,dz - \int_L f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z).$$

The integrals over $L$ cancel with one another. In addition, the sum of the integrals over $\Gamma_R$ and $\gamma_R$ is the integral over $C_R$, and the sum of the integrals over $\Gamma_\rho$ and $\gamma_\rho$ is the integral over $C_\rho$.

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{C_R} f(z)\,dz - \int_\rho^R \frac{r^{-a}e^{-i2\pi a}}{r+1}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z) \tag{3}$$

This is equation (3) in the textbook, the desired result. Applying Cauchy's residue theorem to the two closed contours in Fig. 111 and adding the resulting equations is effectively what we would have gotten by applying the theorem to the closed contour in Fig. 110.

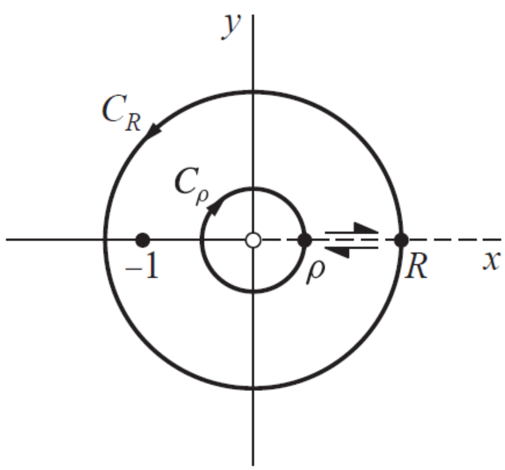


**FIGURE 110**